Lecture 3.

Plan.

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1. Path integral for Green functions in QED.

1.1. Lagrangian QED, gauge invariance.

The Electrodynamics is the theory of electrons and positrons interacting with the electromagnetic field. The electrons and positrons are spin-$\frac{1}{2}$ elementary exitations of Dirac field whose Lagrangian density is

$$\mathcal{L}_D = \bar{\psi}(\imath\gamma^\mu\partial_\mu - m)\psi \tag{1}$$

The elementary exitations of electromagnetic field are the photons, which are spin-1 particles. The Lagrangian density of electromagnetic field is

$$\mathcal{L}_{EM} = -\frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x)$$
$$F_{\mu\nu}(x) = \partial_\mu A_\nu - \partial_\nu A_\mu,\ F^{\mu\nu} = \eta^{\mu\sigma}\eta^{\nu\lambda}F_{\sigma\lambda} \tag{2}$$

The electromagnetic interaction term is given by

$$\mathcal{L}_{int} = -e\bar{\psi}\gamma^\mu A_\mu\psi = J^\mu A_\mu \tag{3}$$

The total Lagrangian can be written in the form

$$\mathcal{L}_{tot} \equiv \mathcal{L}_{EM} + \mathcal{L}_D + \mathcal{L}_{int} = -\frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) + \bar{\psi}(\imath\gamma^\mu D_\mu - m)\psi \tag{4}$$

where

$$D_\mu \equiv \partial_\mu + \imath e A_\mu \tag{5}$$

is so called covariant derivative. The covariant derivative is naturaly arises since the total Lagrangian is invariant under the gauge transformations

$$\psi(x) \to \exp(\imath\alpha(x))\psi(x),\ \ A_\mu \to A_\mu - \frac{1}{e}\partial_\mu\alpha(x) \tag{6}$$

1.2. Green functions of QED in ξ-gauge.

For the ϕ^4 theory we obtained the following path representation of Green's functions:

$$< \Omega|T(\hat{\phi}(\vec{x}_N, t_N)...\hat{\phi}(\vec{x}_1, t_1)|\Omega >= \lim_{T\to\infty} \frac{\int[D\phi]\phi(\vec{x}_1, t_1)...\phi(\vec{x}_N, t_N)\exp[\frac{\imath}{\hbar}\int_{-T}^{T} dtd^3x\mathcal{L}(\phi, \partial\phi)]}{\int[D\phi]\exp[\frac{\imath}{\hbar}\int_{-T}^{T} dtd^3x\mathcal{L}(\phi, \partial\phi)]} \tag{7}$$

Generalizing the corresponding analysis for the case of QED we can write

$$< \Omega|T(\hat{\psi}^{a_N}(x_N)...\hat{\psi}^{a_1}(x_1)\hat{\bar{\psi}}^{b_M}(y_M)...\hat{\bar{\psi}}^{b_1}(y_1)\hat{A}_{\mu_L}(u_L)...\hat{A}_{\mu_1}(u_1)|\Omega >= \lim_{T\to\infty} \frac{\int[D\psi][D\bar{\psi}][DA_\nu]\psi^{a_N}(x_N)...\bar{\psi}^{b_1}(y_1)A_{\mu_L}(u_L)...A_{\mu_1}(u_1)\exp[\frac{\imath}{\hbar}\int_{-T}^{T} dtd^3x\mathcal{L}_{tot}(\psi, A)]}{\int[D\psi][D\bar{\psi}]\exp[\frac{\imath}{\hbar}\int_{-T}^{T} dtd^3x\mathcal{L}_{tot}(\psi, A)]} \tag{8}$$

As we have seen the path integral above is badly determined due to the total action does not change under the gauge transformations and one needs to fix gauge to take into account gauge orbits integration. We considered so called ξ-gauge for the Green's functions of the EM field. It is clear that ξ-gauge can also be used for the Green's functions above if one makes in the expression (8) the replacement

$$\mathcal{L}_{tot} \rightarrow \mathcal{L}_{tot} - \frac{1}{2\xi}(\partial^{\mu} A_{\mu})^2. \tag{9}$$

Of course the expression (8) is not gauge invariant, so we should use Green's functions of gauge invariant operators.

1.3. Interaction representation and Wick theorem.

Modulo the total derivatives the Lagrangian density of EM field in ξ-gauge can be written as a free field Lagrangian:

$$\mathcal{L}_{EM} = -\frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) \approx$$
$$\frac{1}{2}A_{\mu}(\partial^{\nu}\partial_{\nu}A^{\mu} - (1 - \frac{1}{\xi})A_{\mu}\partial^{\mu}\partial_{\nu}A^{\nu}) \tag{10}$$

Thus we have a theory of two free fields interacting to each other due to

$$\mathcal{L}_{int} = -e\bar{\psi}\gamma^{\mu}A_{\mu}\psi = J^{\mu}A_{\mu} \tag{11}$$

with the coupling constant e.

The Green's functions (8) could be calculated by perturbation theory if one could express the Heisenberg operators of fields $\psi(x)$, $\bar{\psi}(x)$, $A_{\mu}(x)$ and vacuum state $|\Omega>$ in terms of the Heisenberg operators of free field of Dirac's fermions, free field of the vector potential and vacuum state $|0>$ of these non interacting fields. In other words, one needs to develope the **interaction picture** for QED.

This can be done in exactly the same way as in the case of ϕ^4 model. As a result we obtain

$$< \Omega|T(\hat{\psi}^{a_N}(x_N)...\hat{\psi}^{a_1}(x_1)\hat{\bar{\psi}}^{b_M}(y_M)...\hat{\bar{\psi}}^{b_1}(y_1)\hat{A}_{\mu_L}(u_L)...\hat{A}_{\mu_1}(u_1)|\Omega >= \lim_{T\to\infty} \frac{< 0|T(\psi^{a_N}(x_N)...\bar{\psi}^{b_1}(y_1)A_{\mu_L}(u_L)...A_{\mu_1}(u_1))\exp[-\imath\int_{-T}^{T}dtH_I(\psi,A)])|0 >}{< 0|T\exp[-\imath\int_{-T}^{T}dtH_I(\psi,A)]|0 >} \tag{12}$$

where

$$H_I(\psi,A) = e\int d^3x\bar{\psi}\gamma^\mu A_\mu\psi \tag{13}$$

The perturbation series appears when we expand the exponentials and get a series of Green's functions of free fields weighted by a powers of e.

The question is **how to calculate these Green's functions?**

For the KG theory and Dirac field theory we know the answer. It is given by Wick theorem. The same theorem is certainly correct for the case at hand:

$$\begin{aligned} T(\psi_I^{a_1}(x_1)...\psi_I^{a_N}(x_N)...\bar{\psi}_I^{b_1}(y_1)...\bar{\psi}_I^{b_M}(y_M)...A_I^{\mu_L}(u_L)) = \\ :\psi_I^{a_1}(x_1)...\psi_I^{a_N}(x_N)...\bar{\psi}_I^{b_1}(y_1)...\bar{\psi}_I^{b_M}(y_M)...A_I^{\mu_L}(u_L): + \\ sum\ of\ :\psi_I^{a_1}(x_1)...\psi_I^{a_N}(x_N)...\bar{\psi}_I^{b_1}(y_1)...\bar{\psi}_I^{b_M}(y_M)...A_I^{\mu_L}(u_L): \\ with\ all\ possible\ contractions\ inside. \end{aligned} \tag{14}$$

The contractions are given by Feynman's propagators:

$$\begin{aligned} S_F^{ab}(x-y) =< 0|T(\psi^a(x)\bar{\psi}^b(y))|0 >= \int \frac{d^4p}{(2\pi)^4}\frac{\imath(p_\mu\gamma^\mu+m)^{ab}}{p^2-m^2+\imath\epsilon}\exp(-\imath p(x-y)), \\ D_F^{\mu\nu}(x-y) =< 0|T(A^\mu(x)A^\nu(y))|0 >= \\ \int\frac{d^4k}{(2\pi)^4}\frac{-\imath}{k^2+\imath\epsilon}(g^{\mu\nu}-(1-\frac{1}{\xi})\frac{k^\mu k^\nu}{k^2})\exp(-\imath k(x-y)) \end{aligned} \tag{15}$$

1.4. An example and Feynman rules.

$$
\begin{aligned}
&< \Omega|T(\psi^a(x)\bar{\psi}^b(y))|\Omega >= \\
&\frac{1}{Z} < 0|T(\psi_I^a(x)\bar{\psi}_I^b(y))|0> + \\
&\frac{-\imath e}{Z} < 0|T(\psi_I^a(x)\bar{\psi}_I^b(y)\int d^4u\bar{\psi}_I(u)\gamma^\mu A_{I\mu}(u)\psi_I(u))|0> + \\
&\frac{(-\imath e)^2}{2!Z} < 0|T(\psi_I^a(x)\bar{\psi}_I^b(y)\int d^4u\bar{\psi}_I(u)\gamma^\mu A_{I\mu}(u)\psi_I(u)\int d^4v\bar{\psi}_I(v)\gamma^\nu A_{I\nu}(v)\psi_I(v))|0> \\
&+...
\end{aligned}
\tag{16}
$$

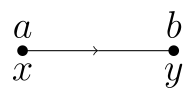
where

$$
Z =< 0|T\exp\left[-\imath e\int d^4(u)\bar{\psi}_I(u)\gamma^\mu A_{I\mu})(u)\psi_I(u)\right]|0> \tag{17}
$$

Let us consider the nominator at $(-\imath e)^0$:

$$
\begin{aligned}
&< 0|T(\psi^a(x)\bar{\psi}^b(y))|0 >= \\
&< 0| : \psi^a(x)\bar{\psi}^b(y) : |0> + < 0|\psi^a(x)\bar{\psi}^b(y)|0 >= S_F^{ab}(x-y).
\end{aligned}
\tag{18}
$$

This term describes free propagation of Dirac field without interaction:

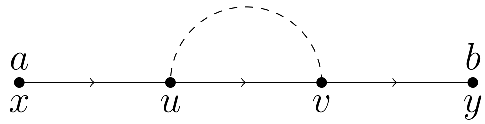
$$
S_F^{ab}(x-y) = \tag{19}
$$


The nominator at first order is equal zero because there is no possibility to contract the field $A_\mu(u)$.

At the second order we have two nontrivial contributions

$$
\begin{aligned}
&< 0|\psi^a(x)\bar{\psi}^b(y)\bar{\psi}(u)\gamma^\mu A_\mu(u)\psi(u)\bar{\psi}(v)\gamma^\nu A_\nu(v)\psi(v)|0> +(u\leftrightarrow v) = \\
&S_F^{ac}(x-u)(\gamma^\mu)_{cd}S_F^{dk}(u-v)(\gamma^\nu)_{kn}S_F^{nb}(v-y)D_{F\mu\nu}(u-v) + (u\leftrightarrow v).
\end{aligned}
\tag{20}
$$

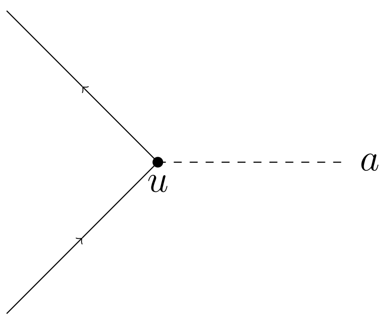
The total coefficient for this contribution is $\frac{(-\imath e)^2}{2!}$. These terms describe the process of propagation of the Dirac field $\psi^a(x)$ from point x to the point u where it emits a photon. Then the fermion propagets to the point v where it absorbs the photon and propagates to the point y:



$$+ (u \leftrightarrow v). \tag{21}$$

We come thereby to the Feynman rules in QED:

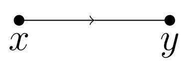
1. Vertex:



$$= -\imath e \int d^4u \gamma_a \tag{22}$$

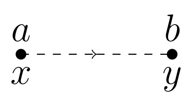
2. Dirac's propagator

$$S_F(x-y) = \tag{23}$$



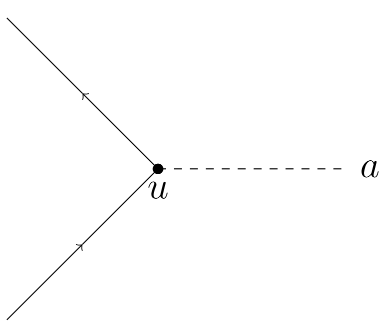
3. Photon's propagator

$$D^F_{ab}(x-y) = \tag{24}$$



The Feynman rules in momenta space:

1. Vertex:



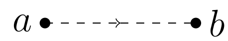
$$= -\imath e \gamma_a \tag{25}$$

2. Dirac's propagator

$$\frac{\imath(p^{\mu}\gamma_{\mu}+m)}{p^2-m^2+\imath\epsilon} = \tag{26}$$

3. Photon's propagator

$$\frac{-\imath\eta_{ab}}{k^2+\imath\epsilon} = \tag{27}$$



where the photon propagator is written in Lorentz gauge $\xi = 1$.

Certainly, the denomonator (17) contains vacuum diagrams which are cancelled with the vacuum diagrams coming from nominator. This phenomenon has already been discussed in ϕ^4.

2. S-matrix and LSZ theorem for QED.

In order to calculate S-matrix elements in QED one needs to detrmine *in* and *out* asymptotic states of photons, electrons and positrons and develope the Feynman rules for the calculation of amplitudes.

2.1. Asymptotic states and LSZ for electrons and positrons.

In the free theory one can create one particle states of electron and positron

$$\begin{aligned} |p,s,+>&= a_s^+(\vec{p})|0>, \\ |p,s,->&= \frac{1}{\sqrt{2E_{\vec{p}}}} b_s^+(\vec{p})|0> \end{aligned} \tag{28}$$

where $\pm$ indicates the value of electric charge and

$$\begin{aligned} a_s^+(\vec{p}) &= \frac{1}{\sqrt{2E_{\vec{p}}}}\int d^3x \exp\left(\imath(E_{\vec{p}}x^0+p_ix^i)\right)\bar{\psi}(x)\gamma^0 u_s(\vec{p}) \\ b_s^+(\vec{p}) &= \frac{1}{\sqrt{2E_{\vec{p}}}}\int d^3x \exp\left(\imath(E_{\vec{p}}x^0+p_ix^i)\right)\bar{v}_s(\vec{p})\gamma^0\psi(x) \end{aligned} \tag{29}$$

where we have used

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\sqrt{E_{\vec{p}}}}\sum_s (a^s_{\vec{p}} u^s(\vec{p})\exp{(-\imath px)} + b^{s\dagger}_{\vec{p}} v^s(\vec{p})\exp{(\imath px)})$$

$$\bar{\psi}(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\sqrt{E_{\vec{p}}}}\sum_s ((a^s_{\vec{p}})^\dagger \bar{u}^s(\vec{p})\exp{(\imath px)} + (b^s_{\vec{p}}) v^s(\vec{p})\exp{(-\imath px)})$$

$$(30)$$

and the relations

$$u^\dagger_s(\vec{p})u_r(\vec{p}) = 2E_{\vec{p}}\delta_{sr},\ \ v^\dagger_s(\vec{p})v_r(\vec{p}) = 2E_{\vec{p}}\delta_{sr},$$
$$u^\dagger_s(\vec{p})v_r(-\vec{p}) = v^\dagger_s(\vec{-}p)u_r(\vec{p}) = 0.$$

$$(31)$$

Let us consider an operator that in the free theory creates a particle with definite spin and charge, localized in momentum space near p_1, and localized in position space near the origin:

$$a^+_{1,s_1} = \int d^3p f_1(\vec{p}) a^+_{s_1}(\vec{p}),\ \ f_1(\vec{p}) \approx \exp{(-\frac{(\vec{p}-\vec{p}_1)}{2\epsilon})}.$$

$$(32)$$

If we time evolve the state created by this timeindependent operator, then the wave packet will propagate. The particle will thus be localized far from the origin as $t \to \pm\infty$. If we consider instead an initial state of the form

$$|(\vec{p}_1, s_1)(\vec{p}_2, s_2 >_{in} = a^+_{1,s_1} a^+_{2,s_2}|0>$$

$$(33)$$

then we have two particles that are widely separated in the far past.

As we stated in Lect. 13, when considering the ϕ^4 theory, **this picture is valid for the interacting theory** (due to Lorentz invariance, locality and claster property of Green's functions).

Although the complication is that a^+_{1,s_1} is time dependent because in QED a^+_{1,s_1} is no longer conserved, in the limit $t \to \infty$ the state

$$|(\vec{p}_1, s_1)(\vec{p}_2, s_2 >_{in} = \lim_{t \to -\infty} a^+_{1,s_1}(t) a^+_{2,s_2}(t)|\Omega > \tag{34}$$

be asymptotic scattering 2-particles in-state (similarly to the ϕ^4 theory).

Analogously the asymtotic scattering 2-particles out-states can be build:

$$|(\vec{p}_1, s_1)(\vec{p}_2, s_2 >_{out} = \lim_{t \to \infty} a^+_{1,s_1}(t) a^+_{2,s_2}(t)|\Omega > \tag{35}$$

We want to find S-matrix element

$$_{out} < (\vec{k}_1, r_1)(\vec{k}_2, r_2)|(\vec{p}_1, s_1)(\vec{p}_2, s_2) >_{in} . \tag{36}$$

In order to calculate this it is helpfull to consider the difference of operators

$$\begin{aligned}
a^+_{1,s_1}(-\infty) - a^+_{1,s_1}(+\infty) = - \int_{-\infty}^{+\infty} dt \partial_t a^+_{1,s_1}(t) = \\
- \int d^3p \frac{f_1(\vec{p})}{\sqrt{2E_{\vec{p}}}} \int d^4x \partial_t \exp{(\imath px)} \bar{\psi}(x) \gamma^0 u_s(\vec{p}) = \\
- \int d^3p \frac{f_1(\vec{p})}{\sqrt{2E_{\vec{p}}}} \int d^4x (\partial_0 \bar{\psi}(x) \gamma^0 - \imath \bar{\psi} \gamma^0 p^0) u_{s_1}(\vec{p}) \exp{(\imath px)} = \\
- \int d^3p \frac{f_1(\vec{p})}{\sqrt{2E_{\vec{p}}}} \int d^4x (\partial_0 \bar{\psi}(x) \gamma^0 - \imath \bar{\psi} \gamma^i p_i - \imath m) u_{s_1}(\vec{p}) \exp{(\imath px)} = \\
\imath \int d^3p \frac{f_1(\vec{p})}{\sqrt{2E_{\vec{p}}}} \int d^4x (\imath \partial_\mu \bar{\psi} \gamma^\mu + m \bar{\psi}) u_{s_1}(\vec{p}) \exp{(\imath px)}
\end{aligned} \tag{37}$$

where in the fourth line we used $(\gamma^\mu p_\mu + m) u_s(\vec{p}) = 0$ and in the last line we integrated by parts.

Notice that in free Dirac's fermions theory the right hand side of this expression is zero since ψ obeys Dirac equation. In QED this is not the case so the scattering is nontrivial.

The hermitian conjugate to (37) is given by

$$a_{1,s_1}(\infty) - a_{1,s_1}(-\infty) = \\ \imath \int d^3p \frac{f_1(\vec{p})}{\sqrt{2E_{\vec{p}}}} \int d^4x \exp(-\imath px)\bar{u}_{s_1}(\vec{p})(-\imath\gamma^\mu\partial_\mu + m)\psi(x) \tag{38}$$

Similarly we find

$$b^+_{1,s_1}(-\infty) - b^+_{1,s_1}(+\infty) = \\ -\imath \int d^3p \frac{f_1(\vec{p})}{\sqrt{2E_{\vec{p}}}} \int d^4x \exp(\imath px)\bar{v}_{s_1}(\vec{p})(-\imath\gamma^\mu\partial_\mu + m)\psi(x) \\ b_{1,s_1}(\infty) - b_{1,s_1}(-\infty) = \\ -\imath \int d^3p \frac{f_1(\vec{p})}{\sqrt{2E_{\vec{p}}}} \int d^4x (\imath\partial_\mu\bar{\psi}\gamma^\mu + m\bar{\psi})v_{s_1}(\vec{p})\exp(-\imath px) \tag{39}$$

Now we can write the matrix element (36) as

$$_{out} < (\vec{k}_1, r_1)(\vec{k}_2, r_2|(\vec{p}_1, s_1)(\vec{p}_2, s_2 >_{in} = \\ < \Omega|a_{2,r_2}(\infty)a_{1,r_1}(\infty)a^+_{1,s_1}(-\infty)a^+_{2,s_2}(-\infty)|\Omega >= \\ < \Omega|T(a_{2,r_2}(\infty)a_{1,r_1}(\infty)a^+_{1,s_1}(-\infty)a^+_{2,s_2}(-\infty))|\Omega > \tag{40}$$

and take the limit $\epsilon \to 0$ so that the wave packets becomes the delta functions. We use (37), (38) to obtain **Lehmann-Symanzik-Zimmermann reduction formula for fermions in QED**

$$_{out} < (\vec{k}_1, r_1)(\vec{k}_2, r_2)|(\vec{p}_1, s_1)(\vec{p}_2, s_2) >_{in} = \\ \frac{\imath^4}{\sqrt{16E_{\vec{k}_1}E_{\vec{k}_2}E_{\vec{p}_1}E_{\vec{p}_2}}} \int d^4x_1 d^4x_2 d^4y_1 d^4y_2 \exp(-\imath k_2 y_2)\exp(-\imath k_1 y_1) \\ < \Omega|T(\bar{u}_{r_2}(\vec{k}_2)(-\imath\gamma^\mu\partial_\mu + m)\psi(y_2)\bar{u}_{r_1}(\vec{k}_1)(-\imath\gamma^\nu\partial_\nu + m)\psi(y_1) \\ (\imath\partial_\lambda\bar{\psi}(x_2)\gamma^\lambda + m\bar{\psi}(x_2))u_{s_2}(\vec{p}_2)(\imath\partial_\tau\bar{\psi}(x_1)\gamma^\tau + m\bar{\psi}(x_1))u_{s_1}(\vec{p}_1))|\Omega > \\ \exp(\imath p_2 x_2)\exp(\imath p_1 x_1) \tag{41}$$

In general the LSZ reduction formula is given by the rules

$$\begin{aligned}
a_s^+(\vec{p})_{in} &\to \frac{\imath}{\sqrt{2E_{\vec{p}}}}\int d^4x(\imath\partial_\mu\bar{\psi}(x)\gamma^\mu + m\bar{\psi})u_s(\vec{p})\exp{(\imath px)} \\
a_s(\vec{p})_{out} &\to \frac{\imath}{\sqrt{2E_{\vec{p}}}}\int d^4x\exp{(-\imath px)}\bar{u}_s(\vec{p})(-\imath\partial_\mu\gamma^\mu + m)\psi(x) \\
b_s^+(\vec{p})_{in} &\to -\frac{\imath}{\sqrt{2E_{\vec{p}}}}\int d^4x\exp{(\imath px)}\bar{v}_s(\vec{p})(-\imath\partial_\mu\gamma^\mu + m)\psi(x) \\
b_s(\vec{p})_{out} &\to -\frac{\imath}{\sqrt{2E_{\vec{p}}}}\int d^4x(\imath\partial_\mu\bar{\psi}(x)\gamma^\mu + m\bar{\psi})v_s(\vec{p})\exp{(-\imath px)}
\end{aligned} \tag{42}$$

It is convenient to rewrite the expression (41) in the momenta representation:

$$\begin{aligned}
{}_{out}<(\vec{k}_1, r_1)(\vec{k}_2, r_2)|(\vec{p}_1, s_1)(\vec{p}_2, s_2)>_{in}= \\
\frac{\imath^4}{\sqrt{16E_{\vec{k}_1}E_{\vec{k}_2}E_{\vec{p}_1}E_{\vec{p}_2}}} \\
<\Omega|T(\bar{u}_{r_2}(-\vec{k}_2)(-\imath\gamma_\mu k_2^\mu + m)\psi(k_2)\bar{u}_{r_1}(-\vec{k}_1)(-\imath\gamma_\nu k_1^\nu + m)\psi(k_1) \\
\bar{\psi}(-p_2)(-\imath\gamma_\lambda p_2^\lambda + m)u_{s_2}(\vec{p}_2)\bar{\psi}(-p_1)(im\gamma_\tau p_1^\tau + m)u_{s_1}(\vec{p}_1))|\Omega>
\end{aligned} \tag{43}$$

Similar to ϕ^4 model the factors like $(-\imath\gamma_\lambda p_1^\lambda + m)$ vanish on-shell. On the other hand the Green's function above has a pole when $(-\imath\gamma_\lambda p_1^\lambda + m) = 0$. As a result we can write

$$\begin{aligned}
S_c((\vec{p}_1, s_1)(\vec{p}_2, s_2)|(\vec{k}_1, r_1)(\vec{k}_2, r_2)) = \\
Z_\psi^2 <\Omega|T(\bar{u}_{r_2}(-\vec{k}_2)\psi(k_2)\bar{u}_{r_1}(-\vec{k}_1)\psi(k_1)\bar{\psi}(-p_2)u_{s_2}(\vec{p}_2)\bar{\psi}(-p_1)u_{s_1}(\vec{p}_1))|\Omega>_{amp}
\end{aligned} \tag{44}$$

2.2. Asymptotic states and LSZ for photons.

It is convenient ot choose Lorentz gauge $\xi = 1$ to write out the asymtotic states for the photons. In this gauge each component of gauge potential obeys KG equation. Therefore we can decompose the quantum EM field in the basis of classical solutions

$$A_\mu(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \sum_{i=0}^{3} \left(a_i(\vec{p})\epsilon^i_\mu(\vec{p}) \exp\left(-\imath px\right) + a_i^+(\vec{p})\epsilon^{*i}_\mu(\vec{p}) \exp\left(\imath px\right)\right) \tag{45}$$

where $i = 0, ..., 3$ number the basis polarization vectors.

The corresponding **LSZ reduction formula for photons in QED** is given by

$$\begin{aligned} a_i^+(\vec{k})_{in} &\to -\frac{\imath}{\sqrt{2E_{\vec{k}}}} \epsilon_i^{*\mu}(\vec{k}) \int d^4x \exp\left(\imath kx\right) \partial_\nu \partial^\nu A_\mu(x) \\ a_i(\vec{k})_{out} &\to -\frac{\imath}{\sqrt{2E_{\vec{k}}}} \epsilon_i^{\mu}(\vec{k}) \int d^4x \exp\left(-\imath kx\right) \partial_\nu \partial^\nu A_\mu(x) \end{aligned} \tag{46}$$

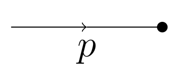
These rules allows to generalize (44) for the case asymptotic states including the photons. The only feature of the asymptotic states of the photons is that the polarization vectors take the form

$$\epsilon_i = (0, \vec{e}),\ \vec{p} \cdot \vec{e} = 0 \tag{47}$$

in order to exclude time like photons.

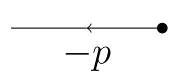
2.3. Feynman rules for scattering processes.

1.For each incoming electron, draw a solid line with an arrow pointed towards the vertex, and label it with the electron's four-momentum, p:

$u^s(p) =$ 

(48)

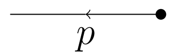
2.For each outgoing electron, draw a solid line with an arrow pointed away from the vertex, and label it with the electron's momentum, p:

$\bar{u}^s(p) =$ 

(49)

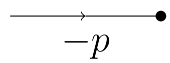
3.For each incoming positron draw a solid line with an arrow pointed away from the vertex, and label it with minus the positron's momentum, $-p$:

$$\bar{v}^s(p) = \tag{50}$$



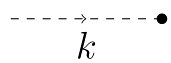
4.For each outgoing positron, draw a solid line with an arrow pointed towards the vertex and label it with minus positron's momentum:

$$v^s(p) = \tag{51}$$



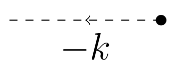
5.For each incoming photon draw a wavy line with an arrow pointed towards the vertex, and label it with the photon's momentum, k:

$$\epsilon^\mu(k) = \tag{52}$$



6.For each outgoing photon draw a wavy line with an arrow pointed away from the vertex, and label it with the photon's momentum, k:

$$\epsilon^{*\mu}(k) = \tag{53}$$



7.The only allowed allowed vertex joins two solid lines, one with an arrow pointing towards it and one with an arrow pointing away from it and one wavy line (whose arrow can point in either direction). Using this vertex join up all the external lines. In this way, draw all possible topologically inequivalent diagrams.

8. Assign each internal line its own momentum. Provide a conservation low for the momenta at each vertex.

9. The value of the diagram consists of the factors: $\epsilon^\mu(k)$ for each incoming photon; $\epsilon^{*\mu}(k)$ for each outgoing photon; $u_s(p)$ for each incoming electron; $\bar{u}_s(p)$ for each outgoing electron; $\bar{v}_s(p)$ for each incoming positron; $v_s(p)$ for each outgoing positron; $-\imath e\gamma^\mu$ for each vertex; $\frac{-\imath\eta_{\mu\nu}}{k^2+\imath\epsilon}$ for each internal photon; $\frac{\imath(p^\mu\gamma_\mu+m)}{p^2-m^2+\imath\epsilon}$ for each internal fermion.